Stats 403 Final Paper: Moment Generating Functions for Univariate and Multivariate Distributions and Their Use in Distribution Theory

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1 Introduction

The Moment Generating Function (MGF) provides a comprehensive summary of a random variable's distribution through its moments. In this paper, we will explore MGFs in different discrete and continuous distributions in the univariate and multivariate cases. Then, we will use the differentiation of the MGF from the definition to find the expected value and variance. In addition, we will introduce the important properties and theorems with their proofs of MGFs and show how to apply them in different scenarios of distribution theory.

2 Univariate Case

Moving into the univariate case, we begin by defining Moment Generating Functions (MGFs), essential for understanding their impact on statistical analysis and distribution theory. This foundational step is critical for later discussions on their properties and applications.

2.1 Definition

The moment generating function (MGF) of a random variable X is defined as

$$M_X(t) = E[e^{tX}] \tag{1}$$

where E denotes the expected value, and t is a real number within the domain where the MGF exists. This function generates the moments of the probability distribution of X by differentiating $M_X(t)$ with respect to t and evaluating at $t = 0$.

2.2 MGFs for Discrete Random Variables

When X is a discrete random variable, its MGF is derived from its p(x) which is probability mass function (pmf) as:

$$M_X(t) = \sum_x e^{tx} p(x) \tag{2}$$

The MGF of discrete random variables is the summation from the range of random variables of the exponential function of the variable, multiplied by its probability mass function. Let's find the MGFs for the discrete distributions of Binomial and Poisson.

2.2.1 MGF of Binomial Distribution

If X follows a Binomial distribution with a total number of n and probability of p such as $X \sim b(n,p)$ where probability mass function of X is $P(X=x) = \binom{n}{x} \cdot p^x(1-p)^{n-x}$, where x = 0,1,...,n The MGF of X is:

$$M_X(t) = E(e^{tx})$$

$$M_X(t) = \sum_x e^{tx} p(x)$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} \cdot p^x(1-p)^{n-x}$$

$$M_X(t) = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x(1-p)^{n-x}$$

By using the binomial theorem $(a+b)^n = \binom{n}{x} \cdot p^k(1-p)^{n-k}$ Let $a = pe^t$ and $b = 1-p$

$$M_X(t) = (pe^t + 1 - p)^n$$

Thus, the MGF of random variables X distributed in the Binomial distribution is $M_X(t) = (pe^t + 1 - p)^n$.

2.2.2 MGF of Poisson Distribution

If X follows a Poisson distribution with a parameter λ where probability mass function of X is $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$ where x = 0,1,... The MGF of X is:

$$M_X(t) = E[e^{tx}]$$

$$M_X(t) = \sum_x e^{tx} p(x)$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$M_X(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Thus, the MGF of random variables X distributed in the Poisson distribution is $M_X(t) = e^{\lambda(e^t-1)}$.

2.3 MGFs for Continuous Random Variables

For a continuous random variable X, the MGF is determined using its f(x) which is the probability density function (pdf) as:

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{3}$$

The MGF of continuous random variables is the integral of the exponential function of the variable, multiplied by its probability density function. Let's find the MGFs for the continuous distributions of Gamma, Exponential, and Normal. In addition, we can find the central and noncentral Chi-Square distribution by MGFs. Let's take a look.

2.3.1 MGF of a Gamma Distribution

If continuous random variables X follows the Gamma distribution, $X \sim \Gamma(\alpha, \beta)$, and the probability density function of X is $f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha}$, $x > 0$, $\alpha > 0$, $\beta > 0$, where $\Gamma(\alpha)$ is the gamma function defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$. Find the moment-generating function of X

$$M_X(t) = E[e^{tx}] = \int_x e^{tx} f(x) \, dx$$

$$M_X(t) = \int_0^\infty e^{tx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} dx$$

$$M_X(t) = \int_0^\infty \frac{x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)}}{\Gamma(\alpha)\beta^\alpha} dx$$

Use the transformation $y = x\left(\frac{1}{\beta} - t\right)$ to get

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

Thus, the MGF of X which follows the Gamma distribution is $(1 - \beta t)^{-\alpha}$.

2.3.2 MGF of Exponential Distribution

If continuous random variable X follows the exponential distribution, $X \sim exp(\lambda)$ and the probability density function of X is $f(x) = \lambda e^{-\lambda x}$. $x \geq 0, \lambda > 0$. The MGF of X is shown as:

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x) \, dx.$$

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx = \lambda \int_0^\infty e^{-x(\lambda - t)} \, dx.$$

This integral converges for $t < \lambda$, and evaluating it, we find

$$M_X(t) = \frac{\lambda}{\lambda - t} = [\frac{\lambda}{\lambda} - \frac{t}{\lambda}]^{-1} = (1 - \frac{t}{\lambda})^{-1}$$

Thus, the MGF of X is $M_X(t) = (1 - \frac{t}{\lambda})^{-1}$, which is a special case of $\Gamma(\alpha, \beta)$ with $\alpha = 1$ and $\beta = \frac{1}{\lambda}$.

2.3.3 MGF of a Normal Distribution

Now, consider X having a normal distribution denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$ with the probability density function of X is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. The MGF of X is shown:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}} dx \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \end{aligned}$$

This simplification shows that the MGF of a normal distribution is $e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

2.3.4 MGF of Chi-Square Distribution

Central Chi-Square

Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with $Z_i \sim N(0, 1)$. If $Y = \sum_{i=1}^n Z_i^2$ then Y follows the chi-square distribution with n degrees of freedom. We write $Y \sim \chi_n^2$.

Proof:

Find the moment generating function of Y. Since $Z_1, Z_2, \ldots, Z_n$ are independent,

$$M_Y(t) = M_{Z_1^2}(t) \times M_{Z_2^2}(t) \times \ldots \times M_{Z_n^2}(t)$$

Each Z_i^2 follows χ_1^2 and therefore it has MGF equal to $(1-2t)^{-\frac{1}{2}}$. Conclusion:

$$M_Y(t) = (1-2t)^{-\frac{n}{2}}.$$

This is the MGF of $\Gamma\left(\frac{n}{2}, 2\right)$, and it is called the chi-square distribution with n degrees of freedom. Its pdf is $f(y) = \frac{y^{\frac{n}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}}$.

Noncentral Chi-Square

Let $Y_1, \ldots, Y_n$ be independent random variables with $Y_i \sim N(\mu_i, \sigma^2)$, $i = 1, \ldots, n$. If each $\mu_i = 0$ then $Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2} \sim \chi_n^2$. What if each $\mu_i \neq 0$?

The moment generating function of $Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2}$ is given by:

$$M_Q(t) = (1-2t)^{-\frac{n}{2}} \exp\left(\frac{t \sum_{i=1}^n \mu_i^2}{\sigma^2(1-2t)}\right).$$

In general, a random variable Q that has an MGF of the form

$$M_Q(t) = (1-2t)^{-\frac{n}{2}} \exp\left(\frac{\theta t}{1-2t}\right)$$

follows the χ^2 distribution with noncentrality parameter θ. We write $Q \sim \chi^2(n, \theta)$. Therefore,

$$Q = \frac{\sum_{i=1}^n Y_i^2}{\sigma^2} \sim \chi^2\left(n, \frac{\sum_{i=1}^n \mu_i^2}{\sigma^2}\right).$$

Note: If the noncentrality parameter is zero then $Q \sim \chi_n^2$ (central χ^2).

2.4 Find Expected Value and Variance by MGFs

To find the k_{th} moment simply evaluate the k_{th} derivative of the $M_X(t)$ at t = 0

$$E[X^k] = [M_X(t)]_{t=0}^{k_{th} derivative} \tag{4}$$

First moment:

$$M_X(t)' = \sum_x xp(x) + \frac{2t}{2!}\sum_x x^2p(x) + \cdots$$

Let t = 0, we see that E(X) = $M_X(0)' = \sum_x xp(x)$
Similarly, to find a second moment

$$M_X(t)'' = \sum_x x^2p(x) + \frac{6t}{3!}\sum_x x^3p(x) + \cdots$$

Let t = 0, we see that $E(X^2) = M_X(0)'' = \sum_x x^2p(x)$ Or from direct differentiation of the MGF from the definition and evaluate the derivatives at t = 0. Also note that $M_X(0) = 1$

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ M_X(t)' &= \frac{\partial}{\partial t}M_X(t) = E[e^{tx}]|_{t=0} = E(X) \\ M_X(t)'' &= \frac{\partial^2}{\partial t^2}M_X(t) = E[x^2e^{tx}]|_{t=0} = E(X^2) \end{aligned}$$

Based on what we got above we can simply got Var(X) by $E(X^2) - (E(X))^2$

2.5 Corollary

Instead of differentiating $M_X(t)$ we can differentiate $ln[M_X(t)]$ and evaluate the first and second derivatives at t=0. This will give E[X] and Var[X]

$$\begin{aligned} \Psi(t) &= \ln[M_X(t)] \\ \Psi'(t) &= \left.\frac{M_X'(t)}{M_X(t)}\right|_{t=0} = \frac{M_X'(0)}{M_X(0)} = E(X) \\ \Psi''(t) &= \left.\frac{M_X''(t)\cdot M_X(t) - [M_X'(t)]^2}{[M_X(t)]^2}\right|_{t=0} = E(X^2) - [E(X)]^2 = \text{Var}(X) \end{aligned}$$

2.6 Properties of MGFs

Now, we can explore the properties of MGFs. There are three important properties of MGFs, let's take a look:

1. If the random variable X plus a constant term a, the moment generating function of (X+a) is denoted as

$$M_{X+a}(t) = E(e^{t(X+a)}) = E(e^{at}) + E(e^{tX}) = e^{at}M_X(t) \tag{5}$$

2. If the random variable X multiplies a constant term b, then the moment-generating function of (bX) would be

$$M_{bX}(t) = E(e^{tbX}) = E(e^{(tb)X}) = M_X(tb) \tag{6}$$

3. If the random variable X plus and a constant term a and divided by a constant term b, the moment-generating function of $(\frac{X+a}{b})$ would be

$$M_{\frac{X+a}{b}}(t) = E(e^{t(\frac{X+a}{b})}) = e^{(\frac{a}{b}t)}M_X(\frac{t}{b}) \tag{7}$$

2.7 Theorems of MGFs

There are two important theorems of MGFs which are about Uniqueness and Independence. Let's take a look at each theorem and its proof.

2.7.1 Uniqueness Theorems

If X and Y are random variables that have the same MGF which is $M_X(t) = M_Y(t)$, then X and Y have the same distribution.

Proof
In this case, X and Y are nonnegative and integer-valued random variables. Let $p_k = P(X = k)$ and $q_k = P(Y = k)$, where k=1,2,... Then $M_X(t) = E[e^{tX}]$ and $M_Y(t) = E[e^{tY}]$
Now, we can find

$$R(t) = M_X[ln(t)] = E[e^{ln(t^x)}] = E(t^x)$$
$$S(t) = M_Y[ln(t)] = E[e^{ln(t^y)}] = E(t^y)$$

If $M_X(t) = M_Y(t)$ it follows that R(t) = S(t) and therefore

$$\sum_{k=1}^{\infty} t^k p_k = R(t) = S(t) = \sum_{k=1}^{\infty} t^t q_k$$

In calculus, if two power series are equal, then their coefficients are also equal. Therefore, $p_k = q_k$, k=1,2,... It follows that X and Y have the same distribution.

2.7.2 Independence of X and Y

Let X, and Y be independent random variables with MGFs $M_X(t)$, $M_Y(t)$ respectively, then the MGF of the sum of these two random variables is equal to the product of the individual MGFs:

$$M_{X+Y}(t) = M_X(t)M_Y(t) \tag{8}$$

Similarly, if apply property 2 above, for independent random variables X and Y, and constants a and b, the MGF of $aX + bY$ is $M_{aX+bY}(t) = M_X(at)M_Y(bt)$

Proof

$$M_{X+Y} = E[e^{t(x+y}] = E[e^{tx}e^{ty}] = E[e^{tx}]E[e^{ty}]$$

because X and Y are independent

$$M_{X+Y} = M_X(t)M_Y(t)$$

These properties and theorems underpin the utility of MGFs in identifying the distributions, calculating moments, and facilitating the analysis of random variables' behavior.

2.8 Applications

1. Use these properties and the moment generating function of $X \sim N(\mu, \sigma)$ to find the moment generating function of $Z \sim N(0, 1)$ to find the moment generating function of $X \sim N(\mu, \sigma)$.

$$X \sim N(\mu, \sigma^2), Z = \frac{X - \mu}{\sigma} \to X = \mu + \sigma Z \sim N(0, 1)$$

$$M_X(t) = M_{\mu+\sigma Z}(t) = E[e^{t(\mu+\sigma Z)}] = e^{t\mu} \cdot E[e^{t\sigma Z}] = e^{t\mu} \cdot M_Z(\sigma t) = e^{t\mu+\frac{1}{2}t^2\sigma^2}$$

2. Suppose X, Y are independent random variables. Find the distribution of $X + Y$, where $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.

$$\begin{aligned} M_{X+Y}(t) &= M_X(t) \cdot M_Y(t) \\ &= e^{t\mu_1+\frac{1}{2}t^2\sigma_1^2} \cdot e^{t\mu_2+\frac{1}{2}t^2\sigma_2^2} \\ &= e^{t(\mu_1+\mu_2)+\frac{1}{2}t^2(\sigma_1^2+\sigma_2^2)} \end{aligned}$$

Thus, $X + Y \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$.

2.9 MGFs for Different Distributions Summations:

2.9.1 MGFs for the sum of binomial distributions.

There are two independent binomial random variables, X and Y, with parameters n_1, p and n_2, p respectively. Find the MGF of their sum, $Z = X + Y$:
The MGF of a binomial random variable X with parameters n, p is given by:

$$M_X(t) = (1 - p + pe^t)^n$$

For two independent binomial random variables X and Y, the MGF of their sum $Z = X + Y$ can be found by multiplying their MGFs due to the independence property:

$$M_Z(t) = M_X(t) \times M_Y(t)$$

Substituting the formulas for $M_X(t)$ and $M_Y(t)$:

$$M_Z(t) = (1 - p + pe^t)^{n_1} \times (1 - p + pe^t)^{n_2}$$

$$M_Z(t) = (1 - p + pe^t)^{n_1+n_2}$$

This shows that Z is indeed a binomial random variable with parameters $n_1 + n_2, p$

2.9.2 MGFs for the sum of Poisson distributions

Consider two independent Poisson random variables X and Y with parameters λ_1 and λ_2 respectively. The MGF of a Poisson random variable with parameter λ is given by:

$$M_X(t) = e^{\lambda(e^t - 1)}$$

The MGF of the sum $Z = X + Y$ for independent Poisson random variables is the product of their individual MGFs:

$$M_Z(t) = M_X(t) \times M_Y(t) = e^{\lambda_1(e^t-1)} \times e^{\lambda_2(e^t-1)}$$

$$M_Z(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

Hence, the sum of two independent Poisson random variables is also a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

2.9.3 MGFs for the sum of Gamma distributions

Let X and Y be two independent gamma random variables with shape parameters α_1, α_2 and a common scale parameter β. The MGF of a gamma random variable is:

$$M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}$$

The MGF of the sum $Z = X + Y$ is given by:

$$M_Z(t) = M_X(t) \times M_Y(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha_1} \times \left(1 - \frac{t}{\beta}\right)^{-\alpha_2}$$

$$M_Z(t) = \left(1 - \frac{t}{\beta}\right)^{-(\alpha_1+\alpha_2)}$$

The sum of two independent gamma random variables with the same scale parameter is a gamma random variable with shape parameter $\alpha_1 + \alpha_2$ and the same scale parameter β.

2.9.4 MGFs for the sum of Normal distributions.

Suppose X and Y are two independent normal random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2 respectively. The MGF of a normal random variable with mean μ and variance σ^2 is:

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

The MGF of the sum $Z = X + Y$ is the product of the MGFs of X and Y:

$$M_Z(t) = M_X(t) \times M_Y(t) = e^{\mu_X t + \frac{1}{2}\sigma_X^2 t^2} \times e^{\mu_Y t + \frac{1}{2}\sigma_Y^2 t^2}$$

$$M_Z(t) = e^{(\mu_X+\mu_Y)t + \frac{1}{2}(\sigma_X^2+\sigma_Y^2)t^2}$$

Thus, the sum of two independent normal random variables is another normal random variable with mean $\mu_X + \mu_Y$ and variance $\sigma_X^2 + \sigma_Y^2$.

3 Multivariate Case

In this section, we take look into the multivariate normal distribution and its moment-generating function (MGF). We will present essential theorems and proofs that highlight the distribution's characteristics and its MGF.

3.1 Definition

Let $X = (X_1, X_2, ..., X_n)'$, be a random vector and let $t = (t_1, t_2, ..., t_n)'$ be a vector of real values. The joint moment generating function of X is defined as:

$$M_X(t) = E(e^{t'X}) = E(e^{\sum_{i=0}^{n} t_i x_i}) \tag{9}$$

Compared to the univariate cases, the MGFs in multivariate have the same equation but the power of exponential has a dot product of t's transpose and X.

3.2 Joint MGF of Multinomial Distribution

Let X be a multinomial distribution which denoted as $X \sim M(n, \mathbf{p})$, where $\mathbf{p}$ is a column vector contain $(p_1, ..., p_r)$ The joint moment generating function of X is:

$$M_X(t) = E[e^{tX}] = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_r} e^{tX}\frac{n!}{x_1!x_2!\ldots x_r!}p_1^{x_1}p_2^{x_2}\ldots p_r^{x_r}$$

$$= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_r}\frac{n!}{x_1!x_2!\ldots x_r!}(p_1e^{t_1})^{x_1}(p_2e^{t_2})^{x_2}\ldots(p_re^{t_r})^{x_r}$$

Using the multinomial theorem we get the joint moment generating function of the multinomial distribution

$$M_X(\mathbf{t}) = (p_1e^{t_1} + p_2e^{t_2} + \ldots + p_re^{t_r})^n \tag{10}$$

Example

Let's show that $U = X_1 + X_2 \sim Binomial(n, p_1 + p_2)$ if $\mathbf{X} \sim$ Multinoimal(n,$\mathbf{p}$).

$$M_{\mathbf{U}}(\mathbf{t}) = M_{X_1,X_2}(\mathbf{t}) = E\left(e^{t\cdot(X_1+X_2)}\right) = \sum\sum\cdots\sum\frac{n!}{x_1!\cdots x_r!}(p_1e^t)^{x_1}(p_2e^t)^{x_2}\cdots p_r^{x_r}$$

$$= (p_1e^t + p_2e^t + \ldots + p_r)^n$$

$$= (p_1e^t + p_2e^t + 1 - p_1 - p_2)^n$$

$$= ((p_1 + p_2)e^t + 1 - (p_1 + p_2))^n$$

Therefore, $U = X_1 + X_2 \sim$ Binomial$(n, p_1 + p_2)$.

3.3 Joint MGF of Multivariate Normal Distribution

Let X be a multivariate normal distribution which is denoted as $X \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is a mean vector and $\boldsymbol{\Sigma}$ is a covariance matrix. The moment generating function $M_X(\mathbf{t})$ is given by:

$$M_{\mathbf{X}}(\mathbf{t}) = E[e^{\mathbf{t}'\mathbf{X}}] = E[e^{\mathbf{t}'(\mathbf{X}-\boldsymbol{\mu})+\mathbf{t}'\boldsymbol{\mu}}]$$

Given that $\mathbf{X}$ is multivariate normal, $(\mathbf{X} - \boldsymbol{\mu})$ is also multivariate normal with mean vector $\mathbf{0}$ and the same covariance matrix $\boldsymbol{\Sigma}$. The quadratic term $\mathbf{t}'(\mathbf{X} - \boldsymbol{\mu})$ follows a normal distribution with mean 0 and variance $\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}$. Thus, we can write:

$$M_{\mathbf{X}}(\mathbf{t}) = e^{\mathbf{t}'\boldsymbol{\mu}} E[e^{\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}]$$

Because $(\mathbf{X} - \boldsymbol{\mu})$ is centered, its expectation is just the exponential of its variance/2, leading to:

$$M_{\mathbf{X}}(\mathbf{t}) = e^{\mathbf{t}'\boldsymbol{\mu}+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}} \tag{11}$$

This is the moment-generating function of the multivariate normal distribution.

3.4 Theorems

There are four important theorems related to Multivariate MGFs. Let's take a look at each and see how to apply them.

3.4.1 Theorem 1 - Expected Value and Variance

Similarly to the univariate case, we can find the expected value and variance in multivariate case of X_i by those steps:

Let $M_i(t) = \frac{\partial M_X(t)}{\partial t_i}$, $M_{ii}(t) = \frac{\partial^2 M_X(t)}{\partial t_i^2}$, and $M_{ij}(t) = \frac{\partial^2 M_X(t)}{\partial t_i \partial t_j}$.
Then, $E[X_i] = M_i(0)$, $E[X_i^2] = M_{ii}(0)$, and $E[X_i X_j] = M_{ij}(0)$.

Corollary:
We can find the mean, variances, and covariances using the logarithm of the joint moment generating function.
Let $\psi(t) = \log M_X(t)$, $\psi_i(t) = \frac{\partial}{\partial t_i}\psi_X(t)$, $\psi_{ii}(t) = \frac{\partial^2}{\partial t_i^2}\psi_X(t)$, and $\psi_{ij}(t) = \frac{\partial^2}{\partial t_i \partial t_j}\psi_X(t)$. Then $EX_i = \psi_i(0)$, $var(X_i) = \psi_{ii}(0)$, and $cov(X_i, X_j) = \psi_{ij}(0)$.
We will take look examples in our later Application section.

3.4.2 Theorem 2 - The Marginal MGFs

Let $\mathbf{X} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$. The marginal moment generating function of $\mathbf{Y}$ ($\mathbf{Z}$) is the moment generating function of $\mathbf{X}$ ignoring the vector $\mathbf{Z}$ ($\mathbf{Y}$). This is expressed as $M_{\mathbf{Y}}(\mathbf{u}) = M_{\mathbf{X}}(\mathbf{u}, 0)$ and $M_{\mathbf{Z}}(\mathbf{v}) = M_{\mathbf{X}}(0, \mathbf{v})$, where $\mathbf{t} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$.

Proof

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) &= E[e^{\mathbf{t}'\mathbf{X}}] = E[e^{\mathbf{u}'\mathbf{Y}+\mathbf{v}'\mathbf{Z}}] \\ &= E\left[e^{(\sum u_i Y_i + \sum v_i Z_i)}\right] \\ \text{Now set all } v_i\text{'s } &= 0 \\ &= E[e^{\mathbf{u}'\mathbf{Y}}] = M_{\mathbf{Y}}(\mathbf{u}) \\ &\equiv M_{\mathbf{X}}(\mathbf{u}, 0) \end{aligned}$$

3.4.3 Theorem 3 - Independence

Let $\mathbf{X}$ be $(\mathbf{Y}, \mathbf{Z})'$. If $\mathbf{Y}$ and $\mathbf{Z}$ are independent then $M_{\mathbf{X}}(t) = M_{\mathbf{Y}}(u) M_{\mathbf{Z}}(v)$.

Proof

$$M_{\mathbf{X}}(t) = E(e^{t'\mathbf{X}}) = E(e^{u'\mathbf{Y}+v'\mathbf{Z}})$$

Because of $\mathbf{Y}$ and $\mathbf{Z}$ are independent

$$= E(e^{u'\mathbf{Y}}) E(e^{v'\mathbf{Z}}) = M_{\mathbf{Y}}(u) \cdot M_{\mathbf{Z}}(v)$$

3.4.4 Theorem 4 - The Joint MGFs

Let $\mathbf{X} = (X_1, \ldots, X_n)'$, $\mathbf{t} = (t_1, \ldots, t_n)'$, and $M_{\mathbf{X}}(\mathbf{t})$ be the joint moment generating function of $\mathbf{X}$. Then

(a) Let $V = \sum_{i=1}^n X_i$ and $W = \sum_{i=1}^n a_i X_i + b$.
Show that $M_V(r) = M_{\mathbf{X}}(r)$ and $M_W(r) = e^{br} M_{\mathbf{X}}(a_1 r, a_2 r, \ldots, a_n r)$.

Proof:

$$\begin{aligned} M_V(r) &= E(e^{rV}) = E\left(e^{r\sum X_i}\right) = E\left(e^{rX_1+\ldots+rX_n}\right) = M_{\mathbf{X}}(r), \quad \mathbf{r} = (r, ..., r)' \\ M_W(r) &= E(e^{rW}) = E\left(e^{r(\sum a_i X_i + b)}\right) = e^{rb} E\left(e^{\sum a_i r X_i}\right) = e^{rb} M_{\mathbf{X}}(a_1 r, \ldots, a_n r) \end{aligned}$$

Note:

$$\begin{aligned} M_{\mathbf{X}}(\mathbf{t}) &= E\left(e^{t_1 X_1 + t_2 X_2 + \ldots + t_n X_n}\right) \\ M_W(r) &= E(e^{rW}) = E\left(e^{r(\sum a_i X_i + b)}\right) = e^{rb} E\left(e^{\sum a_i r X_i}\right) = e^{rb} M_{\mathbf{X}}(a_1 r, \ldots, a_n r) \end{aligned}$$

(b) Let $W = \sum_{i=1}^n a_i X_i + b$ and $U = \sum_{i=1}^n c_i X_i + d$. Then W and U have joint moment generating function given by $M_{W,U}(r,s) = e^{br+ds} M_X(a_1 r + c_1 s, \ldots, a_n r + c_n s)$.

Proof:

$$\begin{aligned} M_{W,U}(r,s) &= E[e^{rW+sU}] \\ &= E[e^{r(\sum a_i X_i + b) + s(\sum c_i X_i + d)}] \\ &= e^{rb+sd} E[e^{(ra_1+sc_1)X_1+\ldots+(ra_n+sc_n)X_n}] \\ &= e^{rb+sd} M_X(ra_1 + sc_1, \ldots, ra_n + sc_n) \end{aligned}$$

3.4.5 Theorem 5 - The Distribution of $AX + c$

Let $X \sim \mathcal{N}(\mu, \Sigma)$ and let A be an $m \times n$ matrix and c an $m \times 1$ vector. Then the random vector $Y = AX + c$ follows a multivariate normal distribution $Y \sim \mathcal{N}(A\mu + c, A\Sigma A')$.

Proof: The moment-generating function of Y can be found by considering the transformation of X and is given by:

$$M_Y(t) = E[e^{t'(AX+c)}] = e^{t'c}E[e^{(A't)'X}]$$

Since X is multivariate normal, we can use its MGF:

$$M_X(A't) = \exp\left((A't)'\mu + \frac{1}{2}(A't)'\Sigma(A't)\right)$$

Substituting back into the MGF of Y, we have:

$$M_Y(t) = e^{t'c}\exp\left((A't)'\mu + \frac{1}{2}(A't)'\Sigma(A't)\right)$$

$$M_Y(t) = \exp\left(t'(A\mu + c) + \frac{1}{2}t^(A\Sigma A')t\right)$$

This is the MGF of a multivariate normal distribution with mean $A\mu + c$ and covariance $A\Sigma A'$, which completes the proof.

3.5 Applications

3.5.1 Marginal MGFs Examples

Consider the multinomial probability distribution $X \sim M(n, p)$ with joint moment generating function

$$M_X(t) = (p_1e^{t_1} + p_2e^{t_2} + \ldots + p_re^{t_r})^n$$

Find the marginal moment generating function of X_1 by **Theorem 2**.

$$M_{X_1}(t_1) = (p_1e^{t_1} + p_2 + p_3 + \ldots + p_r) = (p_1e^{t_1} + 1 - p_1)^n$$

where $t_2 = \ldots = t_r = 0$.

3.5.2 Mean and Variance

Since we know the $M_{X_1}(t_1) = (p_1 e^{t_1} + 1 - p_1)^n$, let's see how to use **Theorem 1** to find $E(X_1)$ and $Var(X_1$. First, let $\psi(t) = \ln(M_X(t)) = \ln(p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_r e^{t_r})$. Then:

$$
\begin{aligned}
\psi_1(t) &= \frac{\partial}{\partial t_1}\psi(t) = \frac{\partial}{\partial t_1}\left(\ln(p_1 e^{t_1} + \ldots + p_r e^{t_r})\right) = \frac{np_1 e^{t_1}}{p_1 e^{t_1} + \ldots + p_r e^{t_r}} \\
\text{At } t = 0, \quad E(X_1) &= \psi_1(0) = \frac{np_1}{1} = np_1 \\
\psi_{11}(t) &= \frac{\partial^2}{\partial t_1^2}\psi(t) = \frac{\partial}{\partial t_1}\left(\frac{np_1 e^{t_1}}{p_1 e^{t_1} + \ldots + p_r e^{t_r}}\right) \\
&= np_1 e^{t_1}\left[\frac{p_1 e^{t_1} + \ldots + p_r e^{t_r} - p_1 e^{t_1}}{(p_1 e^{t_1} + \ldots + p_r e^{t_r})^2}\right] \\
\text{At } t = 0, \quad \text{var}(X_1) &= \psi_{11}(0) = np_1 - np_1^2 = np_1(1 - p_1)
\end{aligned}
$$

4 Conclusion

This paper presented a detailed investigation of the Moment Generating Functions (MGFs) for key univariate and multivariate distributions: Binomial, Poisson, Gamma, Exponential, Normal, both central and noncentral Chi-Square, Multinomial, and Multivariate Normal. The exposition aimed to elucidate the concept and properties of MGFs, and demonstrate their practical utility, especially in deriving mean and variance for diverse distributions. It is hoped that this paper has broadened the reader's understanding of MGFs and their significance in statistical analysis. Thank you for reading my paper.